**BME 7310 Computational Laboratory #3**

Due: 9/22/2023, midnight

**Problem #1.** **Understanding Current:** Conservation of cortical current is governed by the PDE

(1)

where  is the electrical current density. Often  is expressed with respect to tissue potential changes, this can be expressed as the gradient of a scalar potential Φ, and electrical conductivity *s.*

(2)

Hence equation (1) can be recast in terms of Φ as,

(3)

under homogeneous electrical conductivity, it can be written as:

(4)

Your job is to compute *point iterative solutions* of (4) when discretized by center finite differences under the following scenarios:

Given the problem domain and boundary conditions below, compute the solution of Φ using the iterative method below with an initial solution vector of Φ=0 everywhere. Iterate until reaching an *absolute L∞ norm-based error* of successive iterates of less than **1x10-5** and report the number of iterations needed to reach this convergence criterion. Estimate the **spectral radius** of the iteration method during the course of the iterations and **compare with the theoretically expected value**. Plot contours of your solution over the computational domain and report the actual numerical value of Φ for at the point x=0.7 and y=0.7 at convergence.



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cos(y)

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x=y=0.05

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(a) Perform the above simulation with the point iterative method SOR and determine the theoretical optimal . Using this value in the iteration. Be sure to report the solution values requested above in order to verify that your solution is essentially unchanged from previous work that used *Jacobi* and *Gauss Seidel*. Also, perform a series of simulations that enable you to plot iteration count as a function of  to confirm whether the theoretical optimum is the same as that found in practice for this problem. Is the speed up in terms of convergence rate relative to Gauss-Seidel in agreement with theory?

**Solution:**

* Simulation with the point iterative method SOR and determine the theoretical optimal 

First,

1. we computed convergent state with Gauss Seidel, OR
2. use the analytical formula for Gauss Seidel spectral radius

to get the Gauss Seidel spectral radius. Then, we determine *omega* from the spectral radius of Gauss Seidel.

Gauss Spectral Radius: 0.975528

omega = 2 / (1 + sqrt(1 - gauss\_spectral)) = 1.72945381728060

Then we run SOR:

SOR Iterations 48

Spectral Radius 0.726608

* Report the solution values requested above in order to verify that your solution is essentially unchanged from previous work that used *Jacobi* and *Gauss Seidel*

Jacobi Iterations 582

Jacobi V(x==0.7, x==0.7): 0.869710123935

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Gauss Seidel Iterations 322

Gauss Seidel V(x==0.7, x==0.7): 0.869996373963

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SOR Iterations 48

SOR V(x==0.7, x==0.7): 0.870227218371

* Perform a series of simulations that enable you to plot iteration count as a function of  to confirm whether the theoretical optimum is the same as that found in practice for this problem.

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Figure . SOR number of iterations vs, omega. Kahan theorem in Burden & Faires states that SOR can converge only if 0 <ω< 2. Range of omega was limited to between 0.8 – 1.95 since going beyond either limits caused the number of iterations to shoot up extremely large, thus scale up the visualization.

|  |
| --- |
| Clear  % ------------------------------------  %perform a series of simulations that enable you to plot iteration count as a function of omega to confirm whether  % the theoretical optimum is the same as that found in practice for this problem  interations\_v = zeros(1);  itr=0;  for omega=0.8:0.02:1.95  h=0.05;  y=[0:.05:1]';  A=zeros(21,21);  for i=1:21  A(i,21)=1;  A(1,i)=1;  A(21,i)=1;  A(i,1)=cos(2\*pi\*(i\*h-0.05));  end    error=1;  itr=0;  pitr=0;  while (error > 1e-5 & itr < 10000)  itr=itr+1;  Aold=A;  for i=2:20  for j=2:20  A(i,j)=omega \* 1/4 \* ( A(i-1,j) + Aold(i+1,j) + A(i,j-1) + Aold(i,j+1)) ...  + (1-omega)\*Aold(i,j);  end  end    errorold=error;  error=max(max(abs(A-Aold)));  errornew=error;  spectral(itr)=errornew/errorold;  pitr=pitr+1;  if pitr==5  pitr=0;  end  end  if omega==0.8  interations\_v(end)=itr;  else  interations\_v(end+1)=itr;  end  end  figure(5)  plot([0.8:0.02:1.95], interations\_v)  xlabel('omega')  ylabel('SOR num iter')  title('SOR number of iterations w.r.t omega') |

Is the speed up in terms of convergence rate relative to Gauss-Seidel in agreement with theory?

Yes!

SOR has a *theoretical* spectral radius 0.729453817281.

Gauss has a *theoretical* spectral radius 0.975325988997.

Then, the theoretical predicted number of iterations:

N\_iter\_Gauss ~ log10(1e-5)/log10(0. 975325988997) = 460.8209 iterations

N\_iter\_SOR ~ log10(1e-5)/log10(0.729453817281) = 36.4958 iterations.

Thus the speed up agrees with *theoretical* predictions.

**(b)** In previous work, you solved the above using the point interative methods *Jacobi*, *Gauss Seidel*, and now SOR. Specifically, for each method list: *spectral radius, the number of iterations, the computational effort (in terms of number of multiplies and divides) per iteration, and the total computational cost to reach a solution.* Rank the methods according to iteration count and then according to total computational cost. Which method do you comes out on top?

**Answer:**

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| --- | --- | --- | --- | --- |
| **Method** | **Spectral R** | **Num iter** | **Computational effort for iter (for computing domain matrix only)** | **Total computation cost** |
| Jacobi | 0.987764 | 582 | 19 \* 19 (matrix size) | 19 \* 19 \* 582 |
| Gauss | 0.975528 | 322 | 19 \* 19 (2 loops) | 19 \* 19 \* 322 |
| SOR | 0.726608 | 48 | 19 \* 19 (2 loops) | 19 \* 19 \* 48 |

**SOR** wins! in both 1) number of iterations & 2) total computation cost.

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Figure . Contours of the domain for each method.

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Figure . Spectral radius: Jacobi (left), Gauss Seidel (mid), SOR (right).

**Problem #2. Stream Function:**

**(a)** A more interesting problem is shown below for flow around an obstruction. The model has been written in an alternative idealized flow formulation in terms of the **stream function**  which also satisfies Laplace’s equation.

(5)

Use whatever *point iterative* method you prefer and solve the problem below**. Plot contours of your solution and report values at the point x=0.5, y=0.5**. Hint: A *simple way to create the effect of the inner square is to reset =0 at these node positions for each iteration.*

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







x=y=0.05

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

0.4

0.4





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

Type 2 Type 3

**Solution:** Here Gauss Seidel was implemented.

A graph of a number of colored lines

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Figure . Contour and Surface plot of convergence state.

Value at point x=0.5, y=0.5: 1.072052267354

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| figure(1); subplot(1,2,1);subplot(1,2,2);  figure(2);  % Gauss-Seidel  clear;  h=0.05;  x=[0:h:2];  y=[0:h:1];  A=zeros(length(y), length(x));  %% ---- BC -------  A(2:end-1,1) = (1/4) \* ( A(1:end-2, 1) + A(3:end, 1) + 2\*A(2:end-1, 2));  A(2:end-1,end) = 1/(4+2\*h) \* ( A(1:end-2, end) + A(3:end, end) + 2\*A(2:end-1, end-1) - 2\*h); % type3;  A(1,:)=1;  A(end,:)=3;  % ---- obstruction -----  A(6:14,16:24) = 0;    error=1;  itr=0;  pitr=0;  while (error > 1e-5 && itr < 10000)  itr=itr+1;  Aold=A;  for i=2:(length(y)-1)  for j=1:length(x)  if j==1  A(i,j) = 1/4 \* (A(i-1, j) + Aold(i+1, j) + 2\*Aold(i, j+1));  elseif j==length(x)  A(i,j) = 1/(4+2\*h) \* ( A(i-1, j) + Aold(i+1, j) + 2\*A(i, j-1) - 2\*h);  elseif 6<=i && i<=14 && 16<=j && j<=24  A(i,j) = 0;  elseif j~=length(x) && j~=1  A(i,j)=1/4\*(A(i-1,j)+Aold(i+1,j)+A(i,j-1)+Aold(i,j+1));  end  end  end  errorold=error;  error=max(max(abs(A-Aold)));  errornew=error;  spectral(itr)=errornew/errorold;  pitr=pitr+1;  if pitr==5  figure(1);subplot(1,2,1);  contour(A);  pitr=0;  end  end    fprintf('Gauss Seidel Iterations %d\n',itr);  figure(1); subplot(1,2,1);  contour(A);  xlabel('x')  ylabel('y')  title(['GaussSeidel: # of Iterations ' num2str(itr)]);  figure(1); subplot(1,2,2);  [X,Y]=meshgrid(x, y);  surf(X,Y,A)  title(['GaussSeidel: # of Iterations ' num2str(itr)]);  figure(2), clf  plot(spectral);  gauss\_spectral = mean(spectral(itr-5:itr));  fprintf('Spectral Radius %f\n',gauss\_spectral);  fprintf('Value at point x=0.5, y=0.5 %.12f \n', A(10,10)) |

**(b)** For the stream function formulation of ideal flow, the following is true:

 (6)

Using this description, plot the velocity vectors with the ‘quiver’ command. Explain the results you see with respect to plot in part **(a)**. In light of equations (5), and (6) as compared to that of equations (2) and (3) and the respective solutions each produced, each describes ‘flow’, what observations can you make regarding how they differ with respect to the representation of flow?

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Figure . Velocity vector of field.

**Answer:**

* Explain the results you see with respect to plot in part **(a)**

Velocity goes perpendicular to domain contour, and the velocity plot represents changes in space at each position.

If we compare the domain contour plot and ‘velocity plot’, upper part of contour plot presents more rapid change (closer contours with more changes in color), thus bigger vector lengths in velocity plot (faster flow).

* In problem 1, we have 3 Type 1 boundaries and an incoming flow from the left that is perpendicular to the contour, thus create the contour plot as seen.
* In problem 2, we have more flow coming from the right, a gradient/difference of potential from top to bottom of domain, and an obstruction in the middle, thus flow directions like in the velocity plot is seen. The surface plot might illustrate better.
* The velocity ‘potential’ isocontours(problem #1 physics) are orthogonal to the streamlines, i.e. if we were to be solving the potential problem of #1 with the boundary conditions of problem #2, the quiver plot shown above would be crossing potential lines orthogonally in the potential solution. As we can see in problem #2 streamlines, the stream function follows the vectors nicely. So streamlines actually indicate flow direction and where streamlines are concentrated such as at the top of the obstruction, this indicates large flow. More specifically, as opposed to the potential where concentrated contours indicate rapid flow across the contours, concentrated contours in the stream function indicate rapid flow along the contours.

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| --- |
| %%  Vx=zeros(length(y), length(x));  Vy=zeros(length(y), length(x));    Vx(2:end-1, 2:end-1) = ( A(3:end, 2:end-1) - A(1:end-2, 2:end-1) ) ./ (2\*h);  Vy(:, 2:end-1) = - ( A(:,3:end) - A(:,1:end-2) ) ./ (2\*h);  figure(3), clf  quiver(x, y, Vx, Vy)  title('Velocity vector') |