**BME 7310 Computational Laboratory #3**

Due: 9/22/2023, midnight

**Problem #1.** **Understanding Current:** Conservation of cortical current is governed by the PDE

(1)

where  is the electrical current density. Often  is expressed with respect to tissue potential changes, this can be expressed as the gradient of a scalar potential Φ, and electrical conductivity *s.*

(2)

Hence equation (1) can be recast in terms of Φ as,

(3)

under homogeneous electrical conductivity, it can be written as:

(4)

Your job is to compute *point iterative solutions* of (4) when discretized by center finite differences under the following scenarios:

Given the problem domain and boundary conditions below, compute the solution of Φ using the iterative method below with an initial solution vector of Φ=0 everywhere. Iterate until reaching an *absolute L∞ norm-based error* of successive iterates of less than **1x10-5** and report the number of iterations needed to reach this convergence criterion. Estimate the **spectral radius** of the iteration method during the course of the iterations and **compare with the theoretically expected value**. Plot contours of your solution over the computational domain and report the actual numerical value of Φ for at the point x=0.7 and y=0.7 at convergence.



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cos(y)

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x=y=0.05

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**(a)** Perform the above simulation with the point iterative method **SOR and determine the theoretical optimal .** Using this value in the iteration. Be sure to report **the solution values requested above in order to verify that your solution** is essentially unchanged from previous work that used *Jacobi* and *Gauss Seidel*. Also, perform **a series of simulations that enable you to plot iteration count** as a **function of ** to confirm whether the **theoretical optimum** is the same as that found in practice for this problem**. Is the speed up in terms of convergence rate relative to Gauss-Seidel in agreement with theory?**

**Solution:**

* Simulation with the point iterative method SOR and determine the theoretical optimal 

First,

1. we computed convergent state with Gauss Seidel, OR
2. use the analytical formula for Gauss Seidel spectral radius

to get the Gauss Seidel spectral radius. Then, we determine omega from the spectral radius of Gauss Seidel.

Gauss Spectral Radius: 0.975528

omega = 2 / (1 + sqrt(1 - gauss\_spectral)) = 1.72945381728060

Then we run SOR:

SOR Iterations 48

Spectral Radius 0.726608

* Report the solution values requested above in order to verify that your solution is essentially unchanged from previous work that used *Jacobi* and *Gauss Seidel*

Jacobi Iterations 582

Jacobi V(x==0.7, x==0.7): 0.869710123935

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Gauss Seidel Iterations 322

Gauss Seidel V(x==0.7, x==0.7): 0.869996373963

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SOR Iterations 48

SOR V(x==0.7, x==0.7): 0.870227218371

* Perform a series of simulations that enable you to plot iteration count as a function of  to confirm whether the theoretical optimum is the same as that found in practice for this problem.

A graph with a line

Description automatically generated

Figure . SOR number of iterations vs, omega. Kahan theorem in Burden & Faires states that SOR can converge only if 0 <ω< 2. Range of omega was limited to between 0.8 – 1.95 since going beyond either limits caused the number of iterations to shoot up extremely large, thus scale up the visualization.

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| --- |
| Clear  % ------------------------------------  %perform a series of simulations that enable you to plot iteration count as a function of omega to confirm whether  % the theoretical optimum is the same as that found in practice for this problem  interations\_v = zeros(1);  itr=0;  for omega=0.8:0.02:1.95  h=0.05;  y=[0:.05:1]';  A=zeros(21,21);  for i=1:21  A(i,21)=1;  A(1,i)=1;  A(21,i)=1;  A(i,1)=cos(2\*pi\*(i\*h-0.05));  end    error=1;  itr=0;  pitr=0;  while (error > 1e-5 & itr < 10000)  itr=itr+1;  Aold=A;  for i=2:20  for j=2:20  A(i,j)=omega \* 1/4 \* ( A(i-1,j) + Aold(i+1,j) + A(i,j-1) + Aold(i,j+1)) ...  + (1-omega)\*Aold(i,j);  end  end    errorold=error;  error=max(max(abs(A-Aold)));  errornew=error;  spectral(itr)=errornew/errorold;  pitr=pitr+1;  if pitr==5  pitr=0;  end  end  if omega==0.8  interations\_v(end)=itr;  else  interations\_v(end+1)=itr;  end  end  figure(5)  plot([0.8:0.02:1.95], interations\_v)  xlabel('omega')  ylabel('SOR num iter')  title('SOR number of iterations w.r.t omega') |

Is the speed up in terms of convergence rate relative to Gauss-Seidel in agreement with theory?

Yes!

SOR has a *theoretical* spectral radius 0.729453817281.

Gauss has a *theoretical* spectral radius 0.975325988997.

Then, the theoretical predicted number of iterations:

N\_iter\_Gauss ~ log10(1e-5)/log10(0. 975325988997) = 460.8209 iterations

N\_iter\_SOR ~ log10(1e-5)/log10(0.729453817281) = 36.4958 iterations.

Thus the speed up agrees with *theoretical* predictions.

**(b)** In previous work, you solved the above using the point interative methods *Jacobi*, *Gauss Seidel*, and now SOR. Specifically, for each method list: spectral radius, the number of iterations, the computational effort (in terms of number of multiplies and divides) per iteration, and the total computational cost to reach a solution. Rank the methods according to iteration count and then according to toal computational cost. Which method do you comes out on top?

**Solution:**

Rank methods

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **Method** | **Spectral R** | **Num iter** | **Computational effort for iter** | **Total computation cost** |
| Jacobi | 0.987764 |  |  |  |
| Gauss | 0.975528 |  |  |  |
| SOR | 0.726608 |  |  |  |

Jacobi Iterations 582

Spectral Radius 0.987764

Theroretical\_spectral\_r\_Jacobi 0.987662994499

Jacobi V(x==0.7, x==0.7): 0.869710123935

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Gauss Seidel Iterations 322

Spectral Radius 0.975528

Theroretical\_spectral\_r\_Gauss 0.975325988997

Gauss Seidel V(x==0.7, x==0.7): 0.869996373963

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SOR Iterations 48

Spectral Radius 0.726608

Analytical Spectral Radius 0.729453817281

SOR V(x==0.7, x==0.7): 0.870227218371

A screenshot of a graph

Description automatically generated

Figure . Contours of the domain for each method.

A screenshot of a graph

Description automatically generated

Figure . Spectral radius: Jacobi (left), Gauss Seidel (mid), SOR (right).

**Problem #2. Stream Function:**

**(a)** A more interesting problem is shown below for flow around an obstruction. The model has been written in an alternative idealized flow formulation in terms of the **stream function**  which also satisfies Laplace’s equation.

(5)

Use whatever *point iterative* method you prefer and solve the problem below**. Plot contours of your solution and report values at the point x=0.5, y=0.5**. Hint: A *simple way to create the effect of the inner square is to reset =0 at these node positions for each iteration.*

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x=y=0.05

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0.4

0.4

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Type 2 Type 3

**Solution:** Here Gauss Seidel was implemented.

A graph of a number of colored lines

Description automatically generated A colorful graph on a white background

Description automatically generated

Figure . Contour and Surface plot of convergence state.

Value at point x=0.5, y=0.5 1.072052267354

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| figure(1); subplot(1,2,1);subplot(1,2,2);  figure(2);  % Gauss-Seidel  clear;  h=0.05;  x=[0:h:2];  y=[0:h:1];  A=zeros(length(y), length(x));  %% ---- BC -------  A(2:end-1,1) = (1/4) \* ( A(1:end-2, 1) + A(3:end, 1) + 2\*A(2:end-1, 2));  A(2:end-1,end) = 1/(4+2\*h) \* ( A(1:end-2, end) + A(3:end, end) + 2\*A(2:end-1, end-1) - 2\*h); % type3;  A(1,:)=1;  A(end,:)=3;  % ---- obstruction -----  A(6:14,16:24) = 0;    error=1;  itr=0;  pitr=0;  while (error > 1e-5 && itr < 10000)  itr=itr+1;  Aold=A;  for i=2:(length(y)-1)  for j=1:length(x)  if j==1  A(i,j) = 1/4 \* (A(i-1, j) + Aold(i+1, j) + 2\*Aold(i, j+1));  elseif j==length(x)  A(i,j) = 1/(4+2\*h) \* ( A(i-1, j) + Aold(i+1, j) + 2\*A(i, j-1) - 2\*h);  elseif 6<=i && i<=14 && 16<=j && j<=24  A(i,j) = 0;  elseif j~=length(x) && j~=1  A(i,j)=1/4\*(A(i-1,j)+Aold(i+1,j)+A(i,j-1)+Aold(i,j+1));  end  end  end  errorold=error;  error=max(max(abs(A-Aold)));  errornew=error;  spectral(itr)=errornew/errorold;  pitr=pitr+1;  if pitr==5  figure(1);subplot(1,2,1);  contour(A);  pitr=0;  end  end    fprintf('Gauss Seidel Iterations %d\n',itr);  figure(1); subplot(1,2,1);  contour(A);  xlabel('x')  ylabel('y')  title(['GaussSeidel: # of Iterations ' num2str(itr)]);  figure(1); subplot(1,2,2);  [X,Y]=meshgrid(x, y);  surf(X,Y,A)  title(['GaussSeidel: # of Iterations ' num2str(itr)]);  figure(2), clf  plot(spectral);  gauss\_spectral = mean(spectral(itr-5:itr));  fprintf('Spectral Radius %f\n',gauss\_spectral);  fprintf('Value at point x=0.5, y=0.5 %.12f \n', A(10,10)) |

**(b)** For the stream function formulation of ideal flow, the following is true:

 (6)

Using this description, plot the velocity vectors with the ‘quiver’ command. Explain the results you see with respect to plot in part **(a)**. In light of equations (5), and (6) as compared to that of equations (2) and (3) and the respective solutions each produced, each describes ‘flow’, what observations can you make regarding how they differ with respect to the representation of flow?

A graph of a graph of a wave

Description automatically generated with medium confidence

Figure . Velocity vector of field.

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| %%  Vx=zeros(length(y), length(x));  Vy=zeros(length(y), length(x));    Vx(2:end-1, 2:end-1) = ( A(3:end, 2:end-1) - A(1:end-2, 2:end-1) ) ./ (2\*h);  Vy(:, 2:end-1) = - ( A(:,3:end) - A(:,1:end-2) ) ./ (2\*h);  figure(3), clf  quiver(x, y, Vx, Vy)  title('Velocity vector') |